

# Developing map and character 1-form of a singular conformal metric of constant curvature one on a compact Riemann surface

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*with an appendix by Wei Wang and Bin Xu*

**Abstract:** A conformal metric  $g$  of constant positive curvature with finite conical singularities on a compact Riemann surface  $\Sigma$  can be thought of as the pullback of the standard metric on the Riemann sphere by a multi-valued locally univalent meromorphic function  $f$  on  $\Sigma \setminus \{\text{singularities}\}$ , called the *developing map* of the metric. We prove that the developing map is a rational function for such a metric on the Riemann sphere having finite singularities with angles equal to  $2\pi$  times positive integers. We show that when the developing map  $f$  of  $g$  has monodromy in  $U(1)$ , the differential  $\frac{df}{f}$ , called the *character 1-form*, becomes an abelian differential of 3rd kind and satisfies some properties. Conversely, using such an abelian differential of 3rd kind, we can construct a metric on  $\Sigma$  of constant positive curvature with conical singularities, which provides new examples of conformal metrics of constant positive curvature with singularities.

## 1 Introduction

Let  $\Sigma$  be a compact Riemann surface and  $p$  a point on  $\Sigma$ . A conformal metric  $g$  on  $\Sigma$  has a *conical singularity* at  $p$  with the *singular angle*  $2\pi\alpha > 0$  if in a neighborhood of  $p$ ,  $g = e^{2\phi} |dz|^2$ , where  $z$  is a local complex coordinate defined in the neighborhood of  $p$  with  $z(p) = 0$  and  $\phi - (\alpha - 1) \ln |z|$  is continuous in the neighborhood. Let  $p_1, \dots, p_n$  be points on  $\Sigma$  and  $g$  a conformal metric on  $\Sigma$  with conical singularity at  $p_j$  with the singular angle  $2\pi\alpha_j > 0$  for  $j = 1, \dots, n$ . We call that the metric  $g$  represents the divisor  $D := \sum_{j=1}^n (\alpha_j - 1) P_j$ . The Gauss-Bonnet formula says that the integral of the curvature on  $\Sigma$  equals  $2\pi$  times

$$\chi(\Sigma) + \deg D,$$

where  $\chi(\Sigma)$  denotes the Euler number of  $\Sigma$ , and  $\deg D = \sum_j (\alpha_j - 1)$  the degree of the divisor  $D$ . A classical problem is whether there exists a conformal metric on  $\Sigma$  of constant curvature  $K$  representing the divisor  $D$ . If  $K \leq 0$ , then the unique metric exists if and only if the left hand side  $\chi(\Sigma) + \deg D \leq 0$ ; see [MO88, Tr91].

If  $\chi(\Sigma) + \deg D > 0$ , or equivalently  $K > 0$ , the problem turns to be quite subtle and is still open now, except that there are some partial results. Troyanov [Tr89] considered the case of two points on the sphere and proved that the necessary and sufficient condition in this case is  $\alpha_1 = \alpha_2$ . A more general result also due to him [Tr91, Theorem 4] says that there exists a metric of constant positive curvature if

$$0 < \chi(\Sigma) + \deg D < \min \{2, 2 \min \alpha_j\}.$$

Luo and Tian [LT92] proved that the above condition is also necessary and the metric is unique, provided that  $\Sigma$  is a sphere and all singular angles lie in  $(0, 2\pi)$ . In case that  $\Sigma$  is a sphere and the divisor  $D$  contains three points, Eremenko [Er04] and Furuta-Hattori [FH] gives a necessary and sufficient condition for the existence of the metric, which is also unique. In particular, the metric of constant curvature  $K \equiv 1$ , which represents an effective  $\mathbb{Z}$ -divisor supporting at three point, is the pullback under some rational function of the standard metric  $g_{\text{st}} := \frac{4|dw|^2}{(1+|w|^2)^2}$  on the Riemann sphere  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . We at first generalize this corollary due to Eremenko to general effective divisors on the sphere.

**Theorem 1.1.** *A conformal metric of constant curvature one, representing an effective  $\mathbb{Z}$ -divisor on the sphere, must be the pullback under some rational function on the sphere of the standard metric on the Riemann sphere.*

In order to explain the proof of Theorem 1, we need recall Eremenko's idea, which also provides a preparation of notations of Theorem 2. Let  $g$  be a conformal metric of constant curvature one on  $\Sigma$  representing the divisor  $D$ . Denote  $\Sigma^* = \Sigma \setminus \{p_1, \dots, p_n\}$ . Every point  $p$  in  $\Sigma^*$  has a neighborhood  $U_p$  isometric (so conformal) to an open set  $\mathfrak{U}_p$  of the Riemann sphere  $\overline{\mathbb{C}}$ . Denoting by  $\mathfrak{f}_p : U_p \rightarrow \mathfrak{U}_p$  this isometry (conformal map), Eremenko [Er04] observed that  $\mathfrak{f}_p$  can be extended to the whole of  $\Sigma^*$  by analytic continuation, and the extension gives a multi-valued locally univalent meromorphic function  $f$  on  $\Sigma^*$ , whose monodromy belongs to the group  $\text{PSU}(2)$  of orientation-preserving isometries of  $\overline{\mathbb{C}}$  (cf Lemma 2.1). That is, the analytic continuation around a loop in  $\Sigma^*$  of a function element  $\mathfrak{f}$  of  $f$  gives us another one  $(a\mathfrak{f} + b)/(-\bar{b}\mathfrak{f} + \bar{a})$ , where  $a, b$  are two complex numbers determined by the loop, and  $|a|^2 + |b|^2 = 1$ . Hence, the metric  $g$  can be thought of as the pullback

$$g = \frac{4|f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2}$$

under  $f$  of  $g_{\text{st}}$ . Moreover, stimulated by Eremenko's paper [Er04, (2)], we continue to observe in Lemma 3.1 that the Schwarzian  $\{f, z\}$  of  $f$  in a neighborhood  $U_j$  of  $p_j$  with the complex coordinate  $z$  with  $z(p_j) = 0$  has the expression

$$\{f, z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = \frac{c_j}{z^2} + \frac{d_j}{z} + \psi_j(z),$$

where  $c_j = (1 - \alpha_j^2)/2$ ,  $d_j$  are constants and  $\psi_j$  are holomorphic functions in  $U_j$ , dependent on the complex coordinate  $z$ . Since the value of  $c_j$  is independent of the choice of the complex coordinate  $z$ , we say that  $f$  has *regular singularity of weight*  $c_j = (1 - \alpha_j^2)/2$  at  $p_j$ . We now arrive at

**Definition 1.1** Let  $g$  be a conformal metric on  $\Sigma$  of constant curvature one representing the divisor  $D$ . We call a multi-valued locally univalent meromorphic function  $f$  on  $\Sigma^*$  a *developing map* of the metric  $g$  if  $g = f^* g_{\text{st}}$ .

Actually, we can see that the structure of the set of all developing maps for such a metric  $g$  is quite simple from the following

**Lemma 1.1.** *Any two developing maps  $f_1, f_2$  of the metric  $g$  are related by a fractional linear transformation  $\mathfrak{L} \in \text{PSU}(2)$ ,  $f_2 = \mathfrak{L} \circ f_1$ . In particular, any two developing maps of  $g$  have mutually conjugate monodromy in  $\text{PSU}(2)$ . Then we call this conjugate class the monodromy of the metric  $g$ . The space of developing maps of the metric  $g$  has a one-to-one correspondence with the quotient group of  $\text{PSU}(2)$  by the monodromy group of a developing map of  $g$ .*

A multi-valued locally univalent meromorphic function  $h$  on  $\Sigma^*$  is said to be *projective* if any two function elements  $h_1, h_2$  of  $h$  near a point  $p \in \Sigma^*$  are related by a fractional linear transformation  $L \in \text{PGL}(2, \mathbb{C})$ ,  $h_1 = L \circ h_2$ .

**Lemma 1.2.** *Let  $f : \Sigma^* \rightarrow \overline{\mathbb{C}}$  be a projective multi-valued locally univalent meromorphic function, and the monodromy of  $f$  belongs to a maximal compact subgroup of  $\text{PGL}(2, \mathbb{C})$ . If  $f$  has regular singularity of weight  $0 \neq c_j < 1/2$  at  $p_j$  for all  $j = 1, \dots, n$ , then there exists a neighborhood  $U_j$  of  $p_j$  with complex coordinate  $z$  and  $L_j \in \text{PGL}(2, \mathbb{C})$  such that  $z(p_j) = 0$  and  $g_j = L_j \circ f$  has the form  $g_j(z) = z^{\alpha_j}$ , where  $0 < \alpha_j \neq 1$  and  $c_j = (1 - \alpha_j^2)/2$ . Moreover, there exists  $\mathfrak{L} \in \text{PGL}(2, \mathbb{C})$  such that the pullback  $(\mathfrak{L} \circ f)^* g_{\text{st}}$  of the standard metric  $g_{\text{st}}$  by  $\mathfrak{L} \circ f$  is a conformal metric of constant curvature one, which represents the divisor  $D = \sum_j (\alpha_j - 1) P_j$ . In particular, if the monodromy of  $f$  belongs to  $\text{PSU}(2)$ , then the fractional linear transformation  $\mathfrak{L}$  turns out to be the identity map.*

**Remark 1.1.** The developing maps also exist for flat or hyperbolic conformal metrics with finite conical or cusp singularities, and an analogue of Lemmata 1.1 and 1.2 holds.

We call a conformal metric  $g$  on  $\Sigma$  of constant curvature one and with finite conical singularities an *abelian metric* if the monodromy of a developing map of the metric  $g$  belongs to an abelian subgroup of  $\text{PSU}(2)$ . Lemma 1.1 tells us that this definition does not depend on the choice of the developing map. We call an abelian metric *non-trivial* if the monodromy of a developing map of the metric is non-trivial. Hence a trivial abelian metric is a pullback of  $g_{\text{st}}$  under some rational function on  $\Sigma$ . We will show in Lemma 4.1 that each abelian subgroup of  $\text{PSU}(2)$  is contained in some maximal torus  $\mathbb{T}$  of  $\text{PSU}(2)$ . By the theory of compact Lie group,  $\mathbb{T}$  is conjugate to the following special maximal torus

$$\text{U}(1) = \left\{ \text{diag} \left( e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta} \right) : \theta \in \mathbb{R} \right\}.$$

Each fractional transformation in  $\text{U}(1)$  is the multiplication by  $e^{2\sqrt{-1}\theta}$ . Therefore, for a non-trivial abelian spherical metric  $g$  on  $\Sigma$ , by Lemma 1.1 there exists a unique developing map of the metric  $g$ , whose monodromy belongs to  $\text{U}(1)$ .

**Definition 1.2** Let  $g$  be a non-trivial abelian spherical metric on a compact Riemann surface  $\Sigma$ . We call a developing map  $f$  of  $g$  *multiplicative* if the monodromy of  $f$  belongs to  $\text{U}(1)$ . We know that such  $f$  exists uniquely. Hence, the logarithmic differential

$$\omega := d(\log f) = \frac{df}{f}$$

of the multiplicative developing map  $f$  is a (single valued) meromorphic 1-form on  $\Sigma$ , which we call the *character 1-form* of the abelian metric  $g$ . Hence the character 1-form of a non-trivial abelian metric is also uniquely defined.

Let  $g$  be a trivial abelian metric on  $\Sigma$ . That is, there exists a rational function  $f : \Sigma \rightarrow \overline{\mathbb{C}}$  such that  $g = f^* g_{\text{st}}$ . It follows from Lemma 1.1 that any other developing map of  $g$  differs from  $f$  by a composition of a fractional linear transformation in  $\text{PSU}(2)$ . We call by a *character 1-form* of the metric  $g$  the logarithmic differential of a developing map of  $g$ . In this case, the set of character 1-forms of the trivial abelian metric  $g$  has a one-to-one correspondence with the group  $\text{PSU}(2)$ .

**Remark 1.2.** The standard metric  $g_{\text{st}}$  on the Riemann sphere  $\overline{\mathbb{C}}$  is a trivial abelian spherical metric. The set of all developing maps of  $g_{\text{st}}$  is exactly the group  $\text{PSU}(2)$ . Any two developing maps of  $g_{\text{st}}$ , fixing 0 and  $\infty$ ,

respectively, differ by a multiple complex constant with modulus 1. The logarithmic differential of all the developing maps, fixing 0 and  $\infty$ , respectively, give the same character 1-form  $\Theta := d(\log w) = \frac{dw}{w}$ , which has two simple poles of 0 and  $\infty$ . The residues of  $\Theta$  at 0 and  $\infty$  equal 1 and  $-1$ , respectively. The algebraic dual vector field  $X := w \frac{\partial}{\partial w}$  of  $\Theta$  is holomorphic with two simple zeroes of 0 and  $\infty$ . The index of  $X$  equals 1 at 0 and  $\infty$ .  $\Phi(w) = \frac{4|w|^2}{1+|w|^2}$  is a smooth Morse function on  $\overline{\mathbb{C}}$ , whose complex gradient field  $\Phi^w \frac{\partial}{\partial w}$  equals  $X$ . Moreover,  $\Phi$  has only two critical points, which are the minimal point 0 and the maximal point  $\infty$ .

Consider a developing map  $f$  of an abelian metric  $g$  on  $\Sigma$ . Assume that  $f$  is multiplicative if  $g$  is non-trivial. We observe that the character 1-form  $\omega = \frac{df}{f} = d(\log f)$  equals the pullback  $f^* \Theta$  of  $\Theta$  by  $f$ . Denote by  $Y := \frac{f(z)}{f'(z)} \frac{\partial}{\partial z}$  the algebraic dual vector field of  $\omega$ , which is a meromorphic vector field on  $\Sigma$ . Then  $Y$  equals the complex gradient field  $\Psi^z \frac{\partial}{\partial z}$  of the function  $\Psi(z) = \frac{4|f(z)|^2}{1+|f(z)|^2}$ . Using these notations, we state the properties of the character 1-form of a non-trivial abelian metric.

**Theorem 1.2.** *Let  $g$  be an abelian metric representing the divisor  $D = \sum_{j=1}^n (\alpha_j - 1) P_j$  with  $1 \neq \alpha_j > 0$ . Let  $f$  be a developing map of  $g$  and be multiplicative if  $g$  is non-trivial. Let  $\omega = \frac{df}{f}$  be the character 1-form of  $g$ ,  $Y$  the algebraic dual vector field of  $\omega$ , and  $\Psi(z) = \frac{4|f(z)|^2}{1+|f(z)|^2}$ .*

(1) *The set of zeroes of the meromorphic vector field  $Y$  coincides with the extremal point set of the function  $\Psi$ . Each zero of  $Y$  is simple, and  $Y$  vanishes at each point  $p_j$  where  $\alpha_j > 0$  is a non-integer. The set of poles of  $Y$  coincides with the saddle point set of  $\Psi$ . Each pole of  $Y$  is some conical singularity  $p_j$  of the abelian metric  $g$ , where  $\alpha_j$  is an integer greater than 1 and the order of the pole  $p_j$  of  $Y$  equals  $\alpha_j - 1$ .*

(2) *Let  $p_1, \dots, p_J$  be the saddle points of  $\Psi$ ,  $p_{J+1}, \dots, p_n$  the singular extremal points of  $\Psi$ , and  $e_1, \dots, e_S$  the smooth extremal points of  $\Psi$  on  $\Sigma^*$ . Then the canonical divisor of the character 1-form  $\omega$  has form*

$$(\omega) = \sum_{j=1}^J (\alpha_j - 1) P_j - \sum_{k=J+1}^n P_k - \sum_{\ell=1}^S E_\ell.$$

*In particular, each pole of  $\omega$  is simple, i.e.  $\omega$  is an abelian differential of 3rd kind. The residue of  $\omega$  at the pole  $e_\ell$  equals 1 or  $-1$ , where  $e_\ell$  is a minimal or maximal point of  $\Psi$ ; the residue of  $\omega$  at the pole  $p_k$  equals  $\alpha_j$  or  $-\alpha_j$ , where  $p_k$  is a minimal or maximal point of  $\Psi$ . Moreover, the real part of  $\omega$  is exact on  $\Sigma' := \Sigma \setminus \{p_{J+1}, \dots, p_n, e_1, \dots, e_S\}$ ,*

$$2\Re \omega = d(\log |f|^2).$$

(3) *The multiplicative developing map  $f$  on  $\Sigma'$  can be expressed by*

$$f(z) = \exp \left( \int^z \omega \right).$$

*In particular, the local monodromy of  $f$  around each  $p_j$  ( $1 \leq j \leq J$ ) is trivial, and the limit  $\lim_{p \rightarrow p_j} f(p)$  exists and belongs to  $\mathbb{C} \setminus \{0\}$ . If we continue analytically a function element  $\mathfrak{f}$  along a simple and sufficiently small loop winding around  $p_k$  ( $J+1 \leq k \leq n$ ) on counterclockwise, then we obtain the one  $\mathfrak{f} \exp(2\pi\sqrt{-1}\alpha_k)$ . The limit  $\lim_{p \rightarrow p_k} |f(p)|$  exists, and equals 0 or  $+\infty$ , provided  $p_k$  is a minimal or maximal point of  $\Psi(z)$ . It is also the case for  $e_\ell$ .*

Using abelian differentials of 3rd kind with the above properties, we can construct new examples of conformal metrics with constant curvature one and with finite conical singularities.

**Theorem 1.3.** *Let  $\omega$  be an abelian differential of 3rd kind having poles on a compact Riemann surface  $\Sigma$ , whose residues are all nonzero real numbers and whose real part is exact outside the set of poles of  $\omega$ . Then there exists a unique abelian metric  $g$  on  $\Sigma$  such that  $\omega$  is one of character 1-forms of  $g$  and  $g$  can be expressed by  $g = f^* g_{\text{st}}$ , where*

$$f(z) = \exp\left(\int^z \omega\right)$$

*is a multi-valued locally univalent meromorphic function unique up to a complex multiple with unit modulus. Suppose that the canonical divisor of  $\omega$  has form*

$$(\omega) = \sum_{j=1}^J (\alpha_j - 1) P_j - \sum_{k=J+1}^N Q_k,$$

*where  $\alpha_j$  are integers  $> 1$ . Then the divisor  $D$  represented by  $g$  has form*

$$D = \sum_{j=1}^J (\alpha_j - 1) P_j + \sum_{k=J+1}^N \left( |\text{Res}_{Q_k}(\omega)| - 1 \right) Q_k$$

*Moreover,  $g$  is a trivial abelian metric if and only if the integral of  $\omega$  on each loop in  $\Sigma \setminus \{\text{poles of } \omega\}$  are  $2\pi\sqrt{-1}$  times integers. In particular, each residue of  $\omega$  is an integer.*

**Remark 1.3.** *The monodromy of a developing map is not abelian for a hyperbolic conformal metric with finite conical or cusp singularities. Moreover, it is also the case for a flat conformal metric with finite conical singularities except that the metric is smooth on a torus. We leave the proof to the interested readers.*

We conclude the introduction by explaining the organization of this paper. In Section 2, we shall firstly make a detailed exposition of Eremenko's idea on the existence of a developing map  $f$  associated with a conformal metric of constant curvature one with finite conical singularities in Lemma 2.1. Then we prove Lemma 1.1 in this section. We shall, in Lemma 3.1 of Section 3, compute the Schwarzian of a developing map of a conformal metric of constant curvature one representing a divisor  $D$ , and also give the proof of Lemma 1.2 and Theorem 1.1. Moreover, we justify in Proposition 3.1 the truth of an intuition that a conformal metric of constant curvature one is actually smooth at a singularity with conical angle  $2\pi$ . We shall in Section 4 prove Theorems 1.2 and 1.3. As an application of Theorem 1.2, in the final section, we will give an alternative proof to a theorem of Troyanov: *if  $g$  is a conformal metric on the sphere  $\overline{\mathbb{C}}$  of constant curvature one and representing the divisor  $D = (\alpha - 1)P + (\beta - 1)Q$ , where  $\alpha, \beta > 0$ , then  $\alpha = \beta$* . Moreover, we shall give some examples of non-abelian conformal metric of constant curvature one. In the appendix by Wang and Xu, it is proved that no cusp singularity would appear in the geometry of nonnegative curvature.

## 2 Existence of developing maps and their monodromy

**Lemma 2.1.** *Let  $g$  be a conformal metric on a compact Riemann surface  $\Sigma$  of constant curvature one and representing the divisor  $D = \sum_{j=1}^n (\alpha_j - 1) P_j$  with  $\alpha_j > 0$ . Then there exists a multi-valued locally univalent holomorphic map  $f$  from  $\Sigma^* := \Sigma \setminus \{p_1, \dots, p_n\}$  to the Riemann sphere  $\overline{\mathbb{C}}$  such that the monodromy of  $f$  belongs to  $\text{PSU}(2)$  and*

$$g = f^* g_{\text{st}}.$$

*Recall that  $g_{\text{st}} = \frac{4|dw|^2}{(1+|w|^2)^2}$  is the standard metric over  $\overline{\mathbb{C}}$ .*

**Proof.** Denote by  $d(\cdot, \cdot)$  the distance on  $\Sigma$  induced by the metric  $g$ . Choose an arbitrary point  $p$  in  $\Sigma^*$  and fix it. Take a positive number  $r = r_p$  sufficiently small such that  $d(p, \{p_1, \dots, p_n\}) > r$  and there exists a geodesic polar coordinate chart in the open metric ball  $B(p, r) = \{q \in \Sigma : d(p, q) < r\} \subset \Sigma^*$ . Choose a positively oriented orthonormal basis  $\{e^1, e^2\} = \{e_p^1, e_p^2\}$  of the tangent space  $T_p \Sigma$ . Choose an arbitrary point  $\mathfrak{p} \in \overline{\mathbb{C}}$  and fix it. Since the Gauss curvature of  $(B(p, r), g)$  is constant one, by a theorem of Riemann ([Pe06, p.136]), there exists an open metric ball  $\mathfrak{B}(\mathfrak{p}, r)$  in the Riemann sphere  $(\overline{\mathbb{C}}, g_{\text{st}})$  and an orientation-preserving isometry  $\mathfrak{f}_p$  from  $(B(p, r), g)$  onto  $(\mathfrak{B}(\mathfrak{p}, r), g_{\text{st}})$ . Denote  $\mathfrak{e}_p^1 := \mathfrak{f}_*(e^1)$ ,  $\mathfrak{e}_p^2 := \mathfrak{f}_*(e^2)$ , which is also a positively oriented orthonormal basis of  $T_p \overline{\mathbb{C}}$ . Then  $\mathfrak{f}_p$  is a conformal map from  $B(p, r)$  to  $\mathfrak{B}(\mathfrak{p}, r)$ .

Take an arbitrary point  $q$  in  $\Sigma^*$  and a curve  $L : [0, 1] \rightarrow \Sigma^*$  joining  $p$  to  $q$ . Then there exists

$$0 < \delta < \min(d(\gamma([0, 1]), \{p_1, \dots, p_n\}), r_p)$$

such that there exists a geodesic polar coordinate chart in the open metric ball  $B(a, \delta)$  for each point  $a$  on the curve  $L$ . If we divide properly the interval  $0 \leq t \leq 1$  into  $n$  subintervals for sufficiently large  $n$ ,  $0 = \gamma_0 < \gamma_1 < \dots < \gamma_n = 1$ , then the curve  $L$  splits into  $n$  subarcs  $L_1, L_2, \dots, L_n$  with  $L_g$  ( $g = 1, \dots, n$ ) joining  $L(\gamma_{g-1}) =: c_{g-1}$  to  $L(\gamma_g) =: c_g$ . Moreover, if we denote by  $B_0, B_1, \dots, B_n$  the open metric balls with centers  $a = c_0, c_1, \dots, c_n$  and with radius  $\delta$ , then the closed arcs  $L_g$  lie completely in  $B_{g-1}$  for  $g = 1, \dots, n$ . Let  $f_0$  be the restriction to  $B_0$  of the conformal map  $\mathfrak{f}_p : B(p, r_p) \rightarrow \mathfrak{B}(\mathfrak{p}, r_p)$ . Then  $f_0$  is an isometry (conformal map) from  $B_0$  onto  $\mathfrak{B}_0 := \mathfrak{B}(\mathfrak{p}, \delta)$ . Choose a positively oriented orthonormal basis  $\{e_{c_1}^1, e_{c_1}^2\}$  of  $T_{c_1} \Sigma$ . Since  $c_1 \in B_0$ , we denote  $\mathfrak{c}_1 := f_0(c_1) \in \mathfrak{B}_0$ ,  $\mathfrak{e}_{c_1}^1 := (f_0)_*(e_{c_1}^1)$  and  $\mathfrak{e}_{c_1}^2 := (f_0)_*(e_{c_1}^2)$ . Then there exists a unique isometry  $f_1 : B_1 \rightarrow \mathfrak{B}_1 := \mathfrak{B}(\mathfrak{c}_1, \delta)$  such that  $f_1(c_1) = \mathfrak{c}_1$  and  $f_1$  maps  $\{e_{c_1}^1, e_{c_1}^2\}$  to  $\{\mathfrak{e}_{c_1}^1, \mathfrak{e}_{c_1}^2\}$ . Then  $f_1 = f_0$  on  $B_0 \cap B_1$ . Since  $c_2 \in L_2 \subset B_1$ ,  $f_1$  is an analytic continuation of  $f_0$  from point  $c_0$  to  $c_2$  along the arc  $L_0 \cup L_1$ . In this way, we obtain  $f_0, \dots, f_n$ , which are recursively defined on  $B_0, \dots, B_n$  and give an analytic continuation of  $\mathfrak{f}_p$  from  $p$  to  $q$  along the curve  $L$ . Using the same argument as [Si69, pp.13-15], we can show that this analytic continuation is independent of the choice of division points on  $L$ . Moreover, if  $L^*$  is another curve in  $\Sigma^*$  joining  $a$  to  $b$  and homotopic to  $L$ , then the result of doing analytic continuation of  $\mathfrak{f}_p$  along  $L^*$  is the same as along  $L$ . Summing up, we obtain a multi-valued locally isometrical (univalent conformal) map  $f$  from  $(\Sigma^*, g)$  to  $(\overline{\mathbb{C}}, g_{\text{st}})$ .

At last, we prove that all the monodromy of  $f$  belongs to  $\text{PSU}(2)$ . Let  $L : [0, 1] \rightarrow \Sigma^*$  be a closed curve with  $L(0) = L(1) = p$ . We use the notation in the previous paragraph. Recall that  $f_0$  maps  $p = c_0$  to  $\mathfrak{p}$  and  $(f_0)_*$  maps  $\{e^1, e^2\}$  to  $\{\mathfrak{e}_p^1, \mathfrak{e}_p^2\}$ , and  $f_n$  maps  $p = c_n$  to  $\mathfrak{c}_n$  and  $(f_n)_*$  maps  $\{e^1, e^2\}$  to  $\{\mathfrak{e}_{c_n}^1, \mathfrak{e}_{c_n}^2\}$ . Then there exists a unique isometry  $\mathfrak{L} \in \text{PSU}(2)$  of  $(\overline{\mathbb{C}}, g_{\text{st}})$  such that  $\mathfrak{L}(\mathfrak{p}) = \mathfrak{c}_n$  and  $\mathfrak{L}_*$  maps  $\{\mathfrak{e}_p^1, \mathfrak{e}_p^2\}$  to  $\{\mathfrak{e}_{c_n}^1, \mathfrak{e}_{c_n}^2\}$ . Therefore  $f_n = \mathfrak{L} \circ f_0$ .  $\square$

**Proof of Lemma 1.1** Take a point  $p \in \Sigma^*$  and a positively oriented orthonormal basis  $\{e^1, e^2\}$  of  $T_p \Sigma^*$ . Let  $\mathfrak{f}_j$  be a function element of  $f_j$  near  $p$  for  $j = 1, 2$ . Denote  $\mathfrak{p}_j := \mathfrak{f}_j(p)$  and  $\mathfrak{e}_{\mathfrak{p}_j}^k := (\mathfrak{f}_j)_*(e^k)$  for  $j, k = 1, 2$ . Then there exists a unique  $\mathfrak{L} \in \text{PSU}(2)$  such that  $\mathfrak{L}(\mathfrak{p}_1) = \mathfrak{p}_2$  and  $\mathfrak{L}_*$  maps  $\{\mathfrak{e}_{\mathfrak{p}_1}^1, \mathfrak{e}_{\mathfrak{p}_1}^2\}$  to  $\{\mathfrak{e}_{\mathfrak{p}_2}^1, \mathfrak{e}_{\mathfrak{p}_2}^2\}$ . Then we obtain the equality  $\mathfrak{f}_2 = \mathfrak{L} \circ \mathfrak{f}_1$  near  $p$ , which implies  $f_2 = \mathfrak{L} \circ f_1$ . It follows from direct computation that the monodromy of  $f_1$  and  $f_2$  are mutually conjugate.

Given a developing map  $f$  and a fractional linear transformation  $\mathfrak{L} \in \text{PSU}(2)$ , we can see  $\mathfrak{L} \circ f = f$  if and only if there exists a point  $p \in \Sigma^*$  and a functional element  $\mathfrak{f}_p$  near  $p$  such that  $\mathfrak{L} \circ \mathfrak{f}_p$  is another function element of  $f$  near  $p$ . That is,  $\mathfrak{L}$  belongs to the image of the monodromy representation of  $\pi_1(\Sigma^*, p)$  with respect to  $\mathfrak{f}$ . Therefore,  $\text{PSU}(2)$  acts in this way transitively on the set of all developing maps with isotropy group isomorphic to the monodromy group.  $\square$

### 3 The Schwarzian of a developing map

**Lemma 3.1.** *Let  $g$  be a conformal metric of constant curvature one on a compact Riemann surface  $\Sigma$  and  $g$  represent a divisor  $D = \sum_{j=1}^n (\alpha_j - 1) P_j$ , where  $\alpha_j > 0$  for all  $j$ . Suppose that  $f : \Sigma^* = \Sigma \setminus \{p_1, \dots, p_n\} \rightarrow \overline{\mathbb{C}}$  is a developing map of  $g$ . Then the Schwarzian  $\{f, z\}$  of  $f$  equals*

$$\{f, z\} = \frac{1 - \alpha_j^2}{2z^2} + \frac{d_j}{z} + \psi_j(z)$$

in a neighborhood  $U_j$  of  $p_j$  with the complex coordinate  $z$  and  $z(p_j) = 0$ , where  $d_j$  are constants and  $\psi_j$  are holomorphic functions in  $U_j$ , depending on the complex coordinate  $z$ .

**Proof.** If we rewrite the metric  $g = \frac{4|f'(z)|^2 |dz|^2}{(1+|f(z)|^2)^2}$  as  $g = e^{2u} |dz|^2$ , then we find  $u = \log |f'(z)| + \log 2 - \log(1 + |f|^2)$ . The lemma in [Tr89, p.300] tells us that

$$\eta(z) = 2 \left( \frac{\partial^2 u}{\partial z^2} - \left( \frac{\partial u}{\partial z} \right)^2 \right) dz^2$$

defines a *projective connection* compatible with the divisor  $D$ . The interested reader could find in [Tr89] the definition of the projective connection, which we will not use in this paper, however. The compatibility of the projective connection  $\eta$  with the divisor  $D$  in [Tr89, p.300] means that

$$\eta(z) = \left( \frac{1 - \alpha_j^2}{2z^2} + \frac{d_j}{z} + \phi_j(z) \right) dz^2, \quad \phi_j \text{ holomorphic,}$$

where  $z$  is the complex coordinate near  $p_j$ . Since the developing map  $f$  is a projective multi-valued function on  $\Sigma^*$ , its Schwarzian  $\{f, z\}$  with respect to the complex coordinate  $z$  near  $p_j$  is a single valued function of  $z$ . By simple computation we obtain that  $\{f, z\}$  equals exactly  $2 \left( \frac{\partial^2 u}{\partial z^2} - \left( \frac{\partial u}{\partial z} \right)^2 \right)$ .  $\square$

**Proof of Lemma 1.2** Recall the well known fact that every maximal compact group of  $\text{PGL}(2, \mathbb{C})$  is conjugate to the subgroup  $\text{PSU}(2)$ . There exists a fractional linear transformation  $\mathfrak{L}$  such that the monodromy of  $\mathfrak{L} \circ f$  belongs to  $\text{PSU}(2)$ . Hence, we may assume that it is the case for  $f$  without loss of generality.

We firstly show the first statement of Lemma 1.2 that *there exists a neighborhood  $U_j$  of  $p_j$  with complex coordinate  $z$  and some  $L_j \in \text{PGL}(2, \mathbb{C})$  such that  $g_j = L_j \circ f$  has form  $z^{\alpha_j}$* . Since  $f$  has a regular singularity of weight  $c_j < 1/2$ , we could choose a neighborhood  $U_j$  of  $p_j$  and a complex coordinate  $x$  on  $U_j$  such that  $x(p_j) = 0$  and

$$\{f, x\} = \frac{c_j}{x^2} + \frac{d_j}{x} + \phi_j(x),$$

where  $\phi_j(x)$  is holomorphic in  $U_j$ . By [Yo87, p.39, Proposition], in the neighborhood  $U_j$ , there are two linearly independent solutions  $u_0$  and  $u_1$  of the equation

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left( \frac{c_j}{x^2} + \frac{d_j}{x} + \phi_j(x) \right) u = 0$$

with single valued coefficient such that  $f(x) = \frac{u_1(x)}{u_0(x)}$ . Actually, we have  $u_0 = \left( \frac{df}{dx} \right)^{-1/2}$  and  $u_1 = f(x)u_0$ . Moreover, if  $f$  changes projectively

$$f \mapsto \frac{af+b}{cf+d} \quad \text{with} \quad ad - bc = 1,$$

then  $u_0$  and  $u_1$  change linearly

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

and vice versa.

Consider the operator  $L_j := x^2 \frac{d^2}{dx^2} + q_j(x)$  with  $q_j(x) = (c_j + d_j x + x^2 \phi_j(x)) / 2$ . Then both  $u_0$  and  $u_1$  are solutions of the equation  $L_j u = 0$ . Since the monodromy of  $f$  belongs to  $\text{PSU}(2)$ , the cyclic group generated by the local monodromy of the equation  $L_j u = 0$  around the  $x = 0$  is contained in a maximal compact subgroup of  $\text{PGL}(2, \mathbb{C})$  conjugate to  $\text{PSU}(2)$ . Note that the equation  $L_j u = 0$  has regular singularity at 0. We could apply the Frobenius method (cf [Yo87, § 2.5]) to solving it. Choose  $\alpha_j > 0$  such that  $c_j = (1 - \alpha_j^2)/2$ . Then the indicial equation

$$f(s) = s(s-1) + \frac{1 - \alpha_j^2}{4} = 0 \quad (1)$$

of the differential equation  $L_j u = 0$  at  $x = 0$  has roots  $s_0 = \frac{1-\alpha_j}{2}$  and  $s_1 = \frac{1+\alpha_j}{2}$ , and  $s_1 - s_0 = \alpha_j > 0$ . Let  $\sum_{k=0}^{\infty} b_k x^k$  be the power series expansion of  $q_j(x)$ , where  $b_0 = c_j/2$ . Let  $s$  be a parameter. Then  $u(s, x) = x^s \sum_{k=0}^{\infty} c_k(s) x^k$  with  $c_0(s) \equiv 1$  is an solution of  $L_j u = 0$  if and only if the equation

$$f(s+n) c_n + R_n = 0 \quad (\sharp_n)$$

holds for all  $n = 0, 1, 2, \dots$ , where

$$R_0 = 0, \quad \text{and for } n > 0, \quad R_n = R_n(c_1, \dots, c_{n-1}, s) = \sum_{i=0}^{n-1} c_i b_{n-i}.$$

Note that the equation  $(\sharp_0)$  is exactly the indicial equation (1). Since  $f(s_1 + n) \neq 0$  for all  $n \geq 1$ , we find that  $u(s_1, x)$  is a solution of the equation.

*Case 1* Suppose that  $s_1 - s_0 = \alpha_j$  is not an integer. Then by the same reason  $u(s_0, x)$  is another solution, which is linearly independent of  $u(s_1, x)$ . Summing up, we have

$$u(s_0, x) = x^{s_0} (1 + \psi_0) \quad \text{and} \quad u(s_1, x) = x^{s_1} (1 + \psi_1),$$

where both  $\psi_0$  and  $\psi_1$  are holomorphic functions vanishing at 0. Here we take a smaller neighborhood of 0 than  $U_j$  to assure the convergence of the power series defining  $\psi_k$  if necessary. Since both  $u_0(x)$  and  $u_1(x)$  are linear combinations of  $u(s_0, x)$  and  $u(s_1, x)$ ,  $f(x) = u_1(x)/u_0(x)$  equals some fractional linear transform of  $u(s_1, x)/u(s_0, x)$ . For simplicity of notation, we may assume  $f(x) = u(s_1, x)/u(s_0, x)$  equals  $x^{\alpha_j}$  times a holomorphic function  $\phi_j(x)$  with  $\phi_j(0) = 1$ . Therefore, we could choose another complex coordinate  $z = z(x)$  of  $U_j$ , under which  $f = f(z) = z^{\alpha_j}$ .

*Case 2* Suppose that  $m := s_1 - s_0 = \alpha_j$  is an integer  $\geq 2$ .

*Subcase 2.1* If  $R_m = 0$ , we can solve the equation  $(\sharp_n)$  for  $s = s_0$  for all  $n \geq 1$  by choosing  $c_m$  arbitrarily, and obtain another solution  $u(s_0, x)$  linearly independent of  $u(s_1, x)$ . The similar argument as Case 1 completes the proof.

*Subcase 2.2* Suppose  $R_m \neq 0$ . Define

$$u^* = x^{s_0} \sum_{k=0}^{\infty} c_k(s_0) x^k,$$



where  $c_0 = 1$ , the  $c_j$ 's ( $1 \leq j < m$ ) are determined by  $(\sharp_j)$ , while  $c_m$  is arbitrarily fixed, and the  $c_j$ 's ( $j > m$ ) are determined also by  $(\sharp_j)$ . Then the linear combination of  $u^*$  and  $\frac{\partial}{\partial s} u(s, x)|_{s=s_1}$

$$U_0(x) := f'(s_1)u^* - R_m \frac{\partial}{\partial s} u(s, x)|_{s=s_1}$$

is a solution. It should be mentioned that since  $f$  and  $R_n$  are holomorphic with respect to  $s$ , it is also the case for the  $c_n$ 's. Then we correct a typo in [Yo87, p.23] and find the two linearly independent solutions given by

$$\begin{pmatrix} U_0(x) \\ u(s_1, x) \end{pmatrix} = \begin{pmatrix} x^{s_0} & x^{s_1} \log x \\ 0 & x^{s_1} \end{pmatrix} \cdot \begin{pmatrix} f'(s_1) \sum_{k=0}^{\infty} c_k(s_0) x^k - R_m x^m \sum_{k=0}^{\infty} c'_k(s_1) x^k \\ \sum_{k=0}^{\infty} c_k(s_1) x^k \end{pmatrix}.$$

Then the local monodromy of the equation  $L_j u = 0$  at  $x = 0$  is the conjugacy class in  $\mathrm{PGL}(2, \mathbb{C})$  of the matrix

$$M = \begin{pmatrix} 1 & 2\pi\sqrt{-1} \\ 0 & 1 \end{pmatrix}.$$

However, the cyclic group generated by  $M$  is a free abelian group, which has no limit point under the usual topology of  $\mathrm{PGL}(2, \mathbb{C})$ . Contradict the fact that the monodromy of the equation  $L_j u = 0$  is contained in a compact group of  $\mathrm{PGL}(2, \mathbb{C})$ . That is, we rule out Subcase 2.2.

Summing up, we prove the statement where  $\alpha_j$  is an integer  $\geq 2$ . Moreover, we can also see that in this case the local monodromy at  $p_j$  is trivial, i.e.  $p_j$  is an apparent singularity of the equation  $L_j u = 0$  and the multi-valued function  $f$ .

Since  $f$  is locally univalent on  $\Sigma^*$  and has monodromy belonging to  $\mathrm{PSU}(2)$ ,  $f^* g_{\mathrm{st}}$  is a well defined smooth Riemannian metric on  $\Sigma^*$  with constant curvature one. The first statement proved just now implies that this metric has conical singularities at  $p_j$  with angles  $2\pi \alpha_j$ .  $\square$

At a point  $p \in \Sigma$  near which  $f = f(z)$  is univalent holomorphic, we find that the Schwarzian  $\{f, z\}$  is holomorphic, and vice versa (cf [Yo87, Remark, p.44]). Actually, we can prove a more general result.

**Lemma 3.2.** *Let  $U$  be an open disk containing 0 in the complex plane  $\mathbb{C}$  with coordinate  $w$  and  $f$  a projective multi-valued meromorphic function on  $U \setminus \{0\}$  with regular singularity of weight zero at 0. That is,  $\{f, w\}$  equals  $\frac{d}{w}$  plus a holomorphic function  $\phi(w)$ , where both the constant  $d$  and  $\phi(w)$  depend on the coordinate  $w$ . Assume that the subgroup of  $\mathrm{PGL}(2, \mathbb{C})$  generated by the local monodromy of  $f$  at 0 is precompact in  $\mathrm{PGL}(2, \mathbb{C})$ . Then there exists  $\mathfrak{L} \in \mathrm{PGL}(2, \mathbb{C})$  and another complex coordinate  $z$  of  $U$  such that  $\mathfrak{L} \circ f(z) = z$  and  $z(0) = 0$ .*

**Proof.** Use the same argument as Case 2 of the proof of Lemma 1.2. Also note that the indicial equation here has two roots 0 and 1.  $\square$

This lemma has the following geometric consequence

**Proposition 3.1.** *The conic singularity with angle  $2\pi$  of a conformal metric with constant curvature one is actually a smooth point of the metric.*

**Proof of Theorem 1.1** Let  $f$  be a developing map of the metric  $g$  representing an effective  $\mathbb{Z}$ -divisor  $\sum_j n_j P_j$  on the sphere. By Lemma 3.1,  $f$  has regular singularity of weight  $(1 - n_j^2)/2$  at  $p_j$ . By Lemma

2.1, the monodromy of  $f$  belongs to  $\text{PSU}(2)$ . By Lemma 1.2, there exists  $\mathfrak{L}_j \in \text{PGL}(2, \mathbb{C})$  and a complex coordinate  $z$  near  $p$  such that  $\mathfrak{L}_j \circ f$  has the form  $f(z) = z^{n_j+1}$  near  $p_j$ , which implies the local monodromy of  $f$  at  $p_j$  is trivial. Since the sphere is simply connected, the monodromy of  $f$  is trivial, i.e.  $f$  is a single valued meromorphic function outside  $\{p_j\}$ . Moreover,  $f$  can be extended meromorphically onto the whole sphere, i.e.  $f$  is a rational function on the sphere.  $\square$

## 4 Proof of Theorems 1.2 and 1.3

**Lemma 4.1.** *Each abelian subgroup of  $\text{PSU}(2)$  is contained in some maximal torus  $\mathbb{T}$  of  $\text{PSU}(2)$ .*

**Proof.** We only need show that it is the case for the Lie group  $\text{SU}(2)$ . Consider the faithful representation  $\rho$  of  $\text{SU}(2)$  on the natural two-dimensional Hermite inner product space  $V$ . Let  $G$  be an abelian subgroup of  $\text{SU}(2)$ . By taking the closure of  $G$  in  $\text{SU}(2)$ , we may assume that  $G$  is compact without loss of generality. Since  $G$  is abelian and compact, Schur's lemma tells us that the restriction representation  $(\rho|_G, V)$  can be decomposed into the direct sum  $V_1 \oplus V_2$  of two 1-dimensional invariant subspaces  $V_1$  and  $V_2$ .  $\square$

### Proof of Theorem 1.2

(1-2) We show that if a point  $p \in \Sigma^*$  is a zero of  $Y(z) = \frac{f(z)}{f'(z)} \frac{\partial}{\partial z}$ , then  $p$  is simple. We choose a function element  $\mathfrak{f}$  near  $p$ . Since the monodromy of  $f$  belongs to  $\text{U}(1)$ ,  $Y = \frac{\mathfrak{f}(z)}{\mathfrak{f}'(z)} \frac{\partial}{\partial z}$ , which is independent of the choice of the function element  $\mathfrak{f}$  and the complex coordinate  $z$ . Since  $\mathfrak{f}$  is a univalent meromorphic function near  $p$ , there exists  $\mathfrak{L} \in \text{PGL}(2, \mathbb{C})$  and a complex coordinate  $z$  near  $p$  with  $z(p) = 0$  such that  $\mathfrak{L} \circ \mathfrak{f} = z$ . Then  $\mathfrak{f} = \frac{az+b}{cz+d}$  with  $ad-bc=1$  near  $p$ , and  $Y = (az+b)(cz+d) \frac{\partial}{\partial z}$ . It is clear that  $p$  could not be a pole of  $Y$ . Since  $Y = 0$  at  $z(p) = 0$ ,  $bd = 0$ .

*Case 1* Since  $ad-bc=1$  and  $bd=0$ , we assume  $b=0$ ,  $d \neq 0$  in this case. Then  $ad=1$ ,  $\mathfrak{f}(z) = \frac{az}{cz+d}$ . Then  $Y = az(cz+d) \frac{\partial}{\partial z}$  has a simple zero at  $z(p) = 0$ . Hence  $\omega = \frac{dz}{az(cz+d)}$  has residue 1 at  $p$ . Since  $f(0) = 0$ ,  $p$  is a minimal point of  $\Psi$  and  $\Psi(p) = 0$ .

*Case 2* Similarly, when  $d=0$  and  $b \neq 0$ ,  $Y$  also has simple zero at  $p$ ,  $\omega$  has residue  $-1$ ,  $\lim_{q \rightarrow p} |f(q)| = +\infty$  and  $p$  is a maximal point of  $\Psi$  and  $\Psi(p) = 4$ .

We show that each point  $q \in \{p_1, \dots, p_n\}$  must be a simple zero of  $Y$ , provided the conical angle of the metric  $g$  at  $q$  equals  $2\pi s > 0$  and  $s$  is a non-integer. By Lemmata 3.1 and 1.2, we can choose a function element  $\mathfrak{f}$  near  $q$  and a complex coordinate  $z$  near  $q$  such that  $\mathfrak{f} = \frac{az^s+b}{cz^s+d}$  with  $ad-bc=1$ . On the other hand, since the monodromy of  $f$  belongs to  $\text{U}(1)$ , so does the local monodromy of  $\mathfrak{f}$ . Then there exists  $\theta \in \mathbb{R}$  such that

$$e^{2\pi\sqrt{-1}\theta} \mathfrak{f} = e^{2\pi\sqrt{-1}\theta} \frac{az^s+b}{cz^s+d} = \frac{ae^{2\pi\sqrt{-1}\theta} z^s+b}{ce^{2\pi\sqrt{-1}\theta} z^s+d}.$$

This is equivalent to that the following equalities hold,

$$\begin{aligned} ac e^{2\pi\sqrt{-1}\theta} (1 - e^{2\pi\sqrt{-1}\theta}) &= 0, \\ (ad e^{2\pi\sqrt{-1}\theta} + bc) - e^{2\pi\sqrt{-1}\theta} (bc e^{2\pi\sqrt{-1}\theta} + ad) &= 0, \\ bd (1 - e^{2\pi\sqrt{-1}\theta}) &= 0. \end{aligned}$$

Solving the equation, we find that either  $c=b=0$  or  $a=d=0$ . i.e.  $\mathfrak{f}(z)$  equals  $\mu z^s$  ( $\mu \neq 0$ ) or  $\lambda z^{-s}$  ( $\lambda \neq 0$ ). Hence  $Y = \pm s z \frac{\partial}{\partial z}$  has simple zero at  $z(q) = 0$ . Since  $\mathfrak{f}(0)$  equals 0 or  $\infty$ ,  $q$  is an extremal point of  $\Psi$ . Precisely speaking,  $q$  is a minimal or maximal point of  $\Psi$  achieving value 0 or 4 if and only if  $\omega$  has

residue  $s > 0$  or  $-s < 0$  at  $p$ .

Let  $p$  be a singular point of the metric  $g$  with conical angle  $2\pi$  times an integer  $n > 1$ . Then  $f(z) = \frac{az^n + b}{cz^n + d}$  and  $Y = \frac{(az^n + b)(cz^n + d)}{nz^{n-1}} \frac{\partial}{\partial z}$ .

*Case A* Assume  $bd \neq 0$ . Then  $p$  is a pole of  $Y$  with order  $n - 1$  and a zero of  $\omega$  with order  $n - 1$ , and  $\lim_{z \rightarrow p} f(z) = f(0) = b/d \in \mathbb{C} \setminus \{0\}$ . Moreover,  $p$  is a saddle point of  $\Psi$ .

*Case B* Assume  $bd = 0$ . Then it is easy to check that  $p$  is a simple zero of  $Y$ . If  $b = 0$  ( $b \neq 0$ ), then  $\lim_{q \rightarrow p} |f(p)|$  equals 0 or  $+\infty$ , where  $\omega$  has residue  $n$  or  $-n$  and  $\Psi$  achieves the minimal value 0 or the maximal value 4.

(3) The local monodromy property of  $f$  follows from Lemmata 3.1 and 1.2.  $\square$

**Proof of Theorem 1.3** We divide the proof into the following two cases.

*Case 1* Assume that the integral of  $\omega$  at some loop in  $\Sigma' := \Sigma \setminus \{\text{poles of } \omega\}$  does not belong to the set  $2\pi\sqrt{-1}\mathbb{Z}$ . Since  $\Re \omega$  is exact on  $\Sigma'$ , solving the equation  $\omega = \frac{df}{f}$  on  $\Sigma'$ , we obtain a multi-valued locally univalent meromorphic function

$$f(z) = \exp \left( \int^z \omega \right)$$

unique up to a complex multiple with modulus one. Moreover,  $f$  has non-trivial monodromy belonging to  $U(1)$  and  $f^* g_{\text{st}}$  is an non-trivial abelian metric with character 1-form  $\omega$ . Conversely, if  $g$  is an abelian metric such that  $\omega$  is one of its character 1-forms, then there exists a developing map  $\tilde{f}$  of  $g$  such that  $\omega = \frac{d\tilde{f}}{\tilde{f}}$ . Since  $\tilde{f}$  has non-trivial monodromy in  $U(1)$ ,  $g$  is non-trivial. Solving the equation  $\omega = \frac{d\tilde{f}}{\tilde{f}}$  also gives the expression of  $\tilde{f}$ ,

$$\tilde{f}(z) = \exp \left( \int^z \omega \right).$$

Therefore, such abelian metric  $g$  is unique. By the argument in the proof of Theorem 1.2, we find the divisor  $D$  represented by  $g$  equals

$$\sum_{j=1}^J (\alpha_j - 1) P_j + \sum_{k=J+1}^N \left( |\text{Res}_{Q_k}(\omega)| - 1 \right) Q_k.$$

*Case 2* Assume that the monodromy given by  $\omega$  is trivial, i.e. the integral of  $\omega$  at each loop in  $\Sigma' := \Sigma \setminus \{\text{poles of } \omega\}$  belongs to the set  $2\pi\sqrt{-1}\mathbb{Z}$ .  $f^* g_{\text{st}}$  with  $f(z) = \exp \left( \int^z \omega \right)$  is a trivial abelian metric such that  $f$  is one of its developing map and a rational function on  $\Sigma$  and  $\omega$  is one of its character 1-forms. Conversely, if  $g$  is an abelian metric with  $\omega$  one of its character 1-forms, then  $g = \tilde{f}^* g_{\text{st}}$  with  $\tilde{f}(z) = \exp \left( \int^z \omega \right)$ . Moreover,  $\tilde{f}$  is a rational function uniquely determined by  $\omega$  up to a complex multiple with modulus one. Therefore, such abelian metric  $g$  is unique. By the similar argument in the proof of Theorem 1.2, we can show that the effective divisor represented by  $g$  equals  $\sum_{j=1}^J (\alpha_j - 1) P_j + \sum_{k=J+1}^N \left( |\text{Res}_{Q_k}(\omega)| - 1 \right) Q_k$ .  $\square$

## 5 Discussions

As an application of Theorems 1.1 and 1.2, we shall show that if  $g$  is conformal metric on the sphere  $\overline{\mathbb{C}}$  of constant curvature one and representing the divisor  $D = (\alpha - 1)P + (\beta - 1)Q$ , where  $\alpha, \beta > 0$ , then  $\alpha = \beta$ .

*Case 1* We assume that at least one of  $\alpha$  and  $\beta$  is not an integer. Suppose that it is the case for  $\alpha$ . Since the punctured sphere  $\overline{\mathbb{C}} \setminus \{p, q\}$  has the fundamental group isomorphic to  $\mathbb{Z}$ , the metric  $g$  is an abelian

metric. Let  $f$  be one of its developing map. By Lemmata 3.1 and 1.2, the local monodromy of  $f$  at  $p$  is non-trivial. Hence  $g$  is non-trivial. We may assume that  $f$  is multiplicative so that  $\omega = \frac{df}{f}$  is the character 1-form of  $g$ . Theorem 1.2 tells us that  $p$  is a simple pole of  $\omega$  with residue  $\pm\alpha$  and  $q$  is either a simple pole or a zero point of  $\omega$ . If  $q$  is a simple pole too, then the residue equals  $\pm\beta$ . Since the only poles of  $\omega$  are  $p, q$ , we have  $\alpha = \beta$ . We shall rule out the case where  $q$  is a zero point of  $\omega$ . Suppose it is the case. Then,  $\omega$  has only one simple pole in  $\Sigma^*$ , which has residue  $\pm 1$  by Theorem 1.2. It contradicts the fact that  $\alpha \neq 1$ .

*Case 2* Suppose that both  $\alpha$  and  $\beta$  are integers  $\geq 2$ . By Theorem 1.1, each developing map  $f$  of  $g$  is a rational function on  $\overline{\mathbb{C}}$ . We may assume that  $p = 0$  and  $f(p) = 0$  by using suitable fractional linear transformations. Since  $f$  is rational,  $f(z) = z^\alpha$  in some complex coordinate  $z$  near 0, which is a simple pole of  $\omega = \frac{df}{f} = \frac{\alpha dz}{z}$ . Similarly,  $q$  is also a simple pole of  $\omega$ . The residue theorem gives  $\alpha = \beta$ .

**Example 5.1.** Consider on the two-sphere a conformal metric  $g$  with constant curvature one and finite conical singularities  $p_1, \dots, p_n$ . Let the angle at  $p_j$  be  $2\pi\alpha_j$ . If  $n \geq 3$  and each  $\alpha_j$  is a non-integer, then  $g$  is not abelian. Otherwise, by Theorem 1.2, the character 1-form of  $g$  would have at least three poles and have no zeroes. Contradiction.

On a torus there exists a conformal metric  $g$  with constant curvature one and a conical singularity  $p$  with angle  $2\pi\alpha$ , where  $1 < \alpha < 3$ . Then we claim that  $g$  is not abelian. It is true by the similar argument in the previous example if  $\alpha$  is not the integer 2. Suppose that  $g$  is abelian when  $\alpha = 2$ . Then  $p$  should be a simple zero of the character 1-form  $\omega$  by Theorem 1.2. Then  $\omega$  has only one simple pole. Contradict the Residue Theorem.

## Appendix

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Let  $g = e^{2\varphi} |dz|^2$  be a smooth conformal metric in a punctured disk  $D \setminus \{0\}$  with finite area and finite Calabi energy, where the Calabi energy  $E(g)$  of  $g$  is defined by the square integral of the Gauss curvature over  $(D \setminus \{0\}, g)$ . Then X. Chen proved in [Chen98, Theorem 2, p.198] that

$$\lim_{\rho \rightarrow 0} \left( \varphi(re^{i\theta}) + \ln r \right) = -\infty, \quad (2)$$

where  $z = re^{i\theta}$ ,  $r = |z|$ . We call that 0 is a *genuine weak cusp singular point* (cf [Chen98, p.195]) of the conformal metric  $g$  if  $\varphi$  satisfies the following integral condition

$$\liminf_{r \rightarrow 0} \int_0^{2\pi} r \frac{\partial(\varphi + \ln r)}{\partial r} = 0. \quad (3)$$

**Theorem A.1** ([Wang11, Theorem 2]) *If the aforementioned metric  $g = e^{2\varphi} |dz|^2$  has nonnegative Gauss curvature in the punctured disk  $D \setminus \{0\}$ , then 0 could not be its genuine weak cusp singular point.*

**Proof.** The proof of the theorem is essentially the same as Lemma 6 in [Chen99, p.218]. Under the coordinate transformation  $t = \ln r$ ,  $\theta = \theta$ , the punctured disk  $D \setminus \{0\} = \{z = re^{i\theta} : 0 < r < 1, -\pi \leq \theta \leq \pi\}$  is transformed to the infinite cylinder

$$\{(t, \theta) : -\infty < t < 0, -\pi \leq \theta \leq \pi\}.$$

Then the metric  $g = e^{2\varphi} |dz|^2$  has the new expression

$$g = e^{2\psi} |dw|^2, \quad \psi(t + i\theta) = \varphi(re^{i\theta}) + t,$$

under the new complex coordinate  $w = t + i\theta$ . (2) means that  $\lim_{t \rightarrow -\infty} \psi(t + i\theta) = -\infty$  for each  $\theta \in [-\pi, \pi]$ . Then  $\Psi(t) := \int_0^{2\pi} \psi(t + i\theta) d\theta$  diverges to  $-\infty$  as  $u$  goes to  $-\infty$ . Then we could choose  $t_1 < 0$  with  $\Psi'(t_1) > 0$ , i.e.

$$\int_0^{2\pi} \frac{\partial \psi}{\partial t}(t_1 + i\theta) d\theta > 0. \quad (4)$$

Since the metric  $g = e^{2\psi} |dw|^2$  has nonnegative Gauss curvature  $K$  in the cylinder, we have

$$-\Delta_{t,\theta} \psi(t + i\theta) = K e^{2\psi} \geq 0 \quad \text{for all } t < t_1.$$

By the Green formula, integrating the above equation in  $[t, t_1] \times S^1$ , we obtain

$$0 \leq - \int_{[t, t_1] \times S^1} \Delta_{t,\theta} \psi(t + i\theta) dt d\theta = \Psi'(t) - \Psi'(t_1).$$

By (4),

$$\begin{aligned} \liminf_{r \rightarrow 0} \int_0^{2\pi} r \frac{\partial(\varphi + \ln r)}{\partial r}(re^{i\theta}) d\theta &= \liminf_{t \rightarrow -\infty} \int_0^{2\pi} \frac{\partial \psi}{\partial t}(t + i\theta) d\theta \\ &= \liminf_{t \rightarrow -\infty} \Psi'(t) \geq \Psi'(t_1) > 0. \end{aligned}$$

Therefore 0 could not be the genuine weak cusp singularity of the metric  $g$  by definition.  $\square$

**Remark** We call that 0 is a *cusp singularity* of the metric  $g = e^{2\varphi} |dz|^2$  if there holds

$$\lim_{r \rightarrow \infty} \frac{\varphi + \ln r}{\ln r} = 0.$$

It is proved in [CWX12] that 0 is a genuine weak cusp singularity of  $g$  if and only if it is a cusp singularity. Intuitively, we may think of a cusp singularity as a conical one with angle 0.

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